

UNIVERSITY COLLEGE LONDON

All questions may be attempted but only marks obtained on the best four questions will count.
The use of an electronic calculator is not permitted in this examination.

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1201**

ASSESSMENT : **MATH1201B**
PATTERN

MODULE NAME : **Algebra 1**

DATE : **10-May-11**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

All questions may be attempted but only marks obtained on the best four solutions will count.

The use of an electronic calculator is **not** permitted in this examination.

1. (i) If $\mathcal{S} = (q \vee \neg p) \wedge (q \wedge \neg p)$ find a formula equivalent to $\neg \mathcal{S}$ which does not involve \wedge, \vee or \neg .

Without computing truth tables decide which of the following (a), (b) or (c) is correct:

- (a) \mathcal{S} is a tautology ; (b) \mathcal{S} is a contradiction ; (c) \mathcal{S} is neither a tautology nor a contradiction.

- (ii) Find a formula equivalent to \mathcal{T} below which does not involve \exists, \neg, \vee or \wedge .

$$\mathcal{T} = \neg(\exists x) \left(\neg P(x) \wedge \left[(\exists y) \neg R(y) \vee \neg(\exists z) \neg S(z) \right] \right).$$

- (iii) Explain what is meant by saying that a mapping $f : A \rightarrow B$ is *bijective*. Prove that if f is bijective then f is invertible.

- iv) Decompose the following permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 6 & 12 & 14 & 9 & 7 & 11 & 4 & 10 & 5 & 8 & 13 & 1 & 2 & 3 \end{pmatrix}$$

into a product of disjoint cycles and hence compute $\text{ord}(\sigma)$ and $\text{sign}(\sigma)$.

2. Let $\epsilon(r, s)$ be the $n \times n$ matrix given by $\epsilon(r, s)_{ij} = \delta_{ri}\delta_{sj}$ where ' δ ' denotes the Kronecker delta. When $n = 1000$ give a formal proof that

$$\epsilon(r, s)\epsilon(777, t) = \begin{cases} \epsilon(r, t) & \text{if } s = 777 \\ 0 & \text{if } s \neq 777. \end{cases}$$

The elementary matrices $\Delta(r, \alpha)$ and $E(r, s; \lambda)$ ($r \neq s$) are defined by

$$\Delta(r, \alpha) = I_n + (\alpha - 1)\epsilon(r, r) ; E(r, s; \lambda) = I_n + \lambda\epsilon(r, s).$$

When $r \neq s$ express (in terms of I_n , $\epsilon(r, r)$ and $\epsilon(r, s)$) the matrix products

- (i) $E(r, s; \mu)\Delta(r, \alpha)$ and (ii) $\Delta(r, \alpha)E(r, s; \lambda)$.

If $\Delta(r, \alpha)E(r, s; \lambda) = E(r, s; \mu)\Delta(r, \alpha)$ express μ in terms of α, λ .

For the matrix A below, find A^{-1} . Moreover, by first expressing A^{-1} as a product of elementary matrices express A also as a product of elementary matrices.

$$A = \begin{pmatrix} -1 & -4 & 1 \\ -2 & -7 & 2 \\ 2 & 3 & -1 \end{pmatrix}.$$

3. Let V, W be vector spaces over a field \mathbb{F} and let $T : V \rightarrow W$ be a mapping; explain what is meant by saying that T is *linear*.

When T is linear, explain what is meant by

- (a) the kernel, $\text{Ker}(T)$ and
(b) the image, $\text{Im}(T)$.

State and prove a relationship which holds between $\dim \text{Ker}(T)$ and $\dim \text{Im}(T)$.

Find the general solution (over \mathbb{Q}) to the system of linear equations $Ax = b$ where

$$A = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 3 & 3 & -3 & -3 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}.$$

If $T_A : \mathbb{Q}^7 \rightarrow \mathbb{Q}^4$ is the linear mapping $T_A(x) = Ax$ find also

- (i) $\dim \text{Ker}(T_A)$; (ii) a basis for $\text{Ker}(T_A)$; (iii) a basis for $\text{Im}(T_A)$.

4. Let $\{v_1, \dots, v_n\}$ be a subset of a vector space V over a field \mathbb{F} ; explain what is meant by saying that the set $\{v_1, \dots, v_n\}$ is linearly independent over \mathbb{F} .

In each case below, decide with justification whether the given vectors are linearly independent over \mathbb{Q} . If they are not, give an explicit dependence relation between them.

(a) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix};$

(b) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix}.$

Explain what is meant by a *spanning set* for a vector space V . Let $\{v_1, v_2, \dots, v_n\}$ be a spanning set for V ($n \geq 3$) and let $\{u_1, u_2\} \subset V$ be a linearly independent set. Show that

- (i) V has a spanning set of the form $\{u_1, v'_2, v'_3, \dots, v'_n\}$ and also that
 (ii) V has a spanning set of the form $\{u_1, u_2, v''_3, \dots, v''_n\}$
 where $v'_i, v''_j \in \{v_1, v_2, \dots, v_n\}$ for each i, j .

5. Let V be the vector space consisting of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$f(x) = a_1 \cos(x) + a_2 \sin(x) + a_3 x \cos(x) + a_4 x \sin(x) \quad (a_i \in \mathbb{Q})$$

and let $D : V \rightarrow V$ be the linear map $D(f) = \frac{df}{dx}$.

Taking $\{\cos(x), \sin(x), x \cos(x), x \sin(x)\}$ as basis for V find :

- i) the matrix of D ; ii) the matrix of D^4 ; iii) the matrix of D^{-1} .

Hence *without further explicit differentiation or integration* write down

- iv) $\frac{d^4}{dx^4} (\sin(x) - 2x \cos(x) + x \sin(x))$ v) $\int \{\cos(x) + x \cos(x) - 2x \sin(x)\} dx$
 [You may ignore the constant of integration in v)].

6. Let $T : U \rightarrow V$ be a linear map between vector spaces U, V , and let $\mathcal{E} = (e_i)_{1 \leq i \leq m}$ be a basis for U and $\Phi = (\varphi_j)_{1 \leq j \leq n}$ be a basis for V .

Explain what is meant by the matrix $\mathcal{M}(T)_{\mathcal{E}}^{\Phi}$ of T taken with respect to \mathcal{E} (on the left) and Φ (on the right).

If $S : V \rightarrow W$ is also linear and $\Psi = (\psi_k)_{1 \leq k \leq p}$ is a basis for W prove that

$$\mathcal{M}(S \circ T)_{\Psi}^{\Phi} = \mathcal{M}(S)_{\Psi}^{\Phi} \mathcal{M}(T)_{\mathcal{E}}^{\Phi}.$$

Let $T : \mathbb{Q}^3 \rightarrow \mathbb{Q}^3$ be the mapping $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & -x_2 & \\ -x_1 & +x_2 & -x_3 \\ x_1 & +x_2 & +2x_3 \end{pmatrix}$

and let $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\Phi = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Write down (i) $\mathcal{M}(T)_{\mathcal{E}}^{\mathcal{E}}$ and (ii) $\mathcal{M}(\text{Id})_{\Phi}^{\mathcal{E}}$. Hence find $\mathcal{M}(T)_{\Phi}^{\Phi}$.